INTRODUCTION TO KÄHLER DIFFERENTIALS

NAVID ALAEI

Abstract. The goal of this project is to provide a brief introduction to the algebraic theory of Kähler differentials. The layout of this project is as follows. We begin with a motivation from geometry. In section 2 we recall some standard results concerning tensor products. In section 3, we introduce the notion of a derivation and the module of Kähler differentials. As a main goal, we work towards establishing a connection between the notion of a tangent/cotangent space of a local ring R and that of a module of Kähler differentials associated to R.

1. Introduction

To motivate the algebraic notion of differentials, let us begin with a more intuitive geometric notion; namely, that of a tangent space. First, we introduce some notation. Let k be a field and recall that the n dimensional affine space over k, denoted $\mathbb{A}_k^n$, is the set of ordered n-tuples with values in k. That is

$$\mathbb{A}_k^n := \{(a_1, \dots, a_n) : a_1, \dots, a_n \in k\}.$$

As usual, we write $R := k[x_1, \dots, x_n]$ to denote the polynomial ring in indeterminates $x_1, \dots, x_n$. Given an ideal I of R, we write $\mathbb{V}(I)$ to mean the vanishing set of I; namely

$$\mathbb{V}(I) := \{(a_1, \dots, a_n) \in \mathbb{A}_k^n : f(a_1, \dots, a_n) = 0 \text{ for all } f \in I\} \subset \mathbb{A}_k^n.$$

As usual, we say that a subset X of $\mathbb{A}_k^n$ is an *affine algebraic set* if $X = \mathbb{V}(I)$ for some ideal I of R. Similarly, if X is an algebraic set, the *ideal* of X is defined as the set of all polynomials in R vanishing on X; i.e.,

$$\mathbb{I}(X) := \{f(x_1, \dots, x_n) \in R : f(a_1, \dots, a_n) = 0 \text{ for all } (a_1, \dots, a_n) \in X\}.$$

Note that the Hilbert Basis Theorem implies that all ideals of R are finitely generated. Further, an affine algebraic set X is called an *affine variety* if $\mathbb{I}(X) \subset R$ is prime. Given an affine variety X, we write

$$k[X] := R/\mathbb{I}(X),$$

for the *coordinate ring* of X. Intuitively, one should think of $k[X]$ as the ring of polynomial functions on X. As $k[X]$ is a domain, we may construct its field of fractions, denoted $k(X)$. This is simply the set of quotients of polynomial functions on X. Next, we have the notion of *regular* functions on an open subset U of X. Given a point $P = (a_1, \dots, a_n) \in X$, we define

$$\mathcal{O}_{X,P} := \left\{ \frac{f}{g} : f, g \in k[X], g(P) \neq 0 \right\} \subset k(X).$$

This is precisely the set of all rational functions that are regular (defined) at P. It is now natural to define $\mathcal{O}_X(U) := \bigcap_{P\in U} \mathcal{O}_{X,P}$. Note $\mathcal{O}_{X,P}$ is a local ring. In fact, the set

$$\mathfrak{m}_P := \{f \in k[X] : f(P) = 0\} = \left\{\frac{f}{g} : f, g \in k[X], f(P) = 0, g(P) \neq 0\right\}$$

is an ideal of $k[X]$, which is maximal. Indeed, $\mathfrak{m}_P$ is the kernel of the evaluation (at P) map $k[X] \to k$. As this map is clearly surjective (all constants functions map to themselves), we obtain, from Noether's First Isomorphism theorem, an isomorphism $\mathcal{O}_{X,P}/\mathfrak{m}_P \cong k$. Further, it is straightforward that any element in $\mathcal{O}_{X,P} \setminus \mathfrak{m}_P$ is invertible, and thus $\mathcal{O}_{X,P}$ is a local ring with maximal ideal $\mathfrak{m}_P$.

We are now ready to define the tangent space to an affine variety at a point. Let X be an affine variety with $P \in X$. After a change of coordinates, we may assume P is the origin. Intuitively, the tangent space to X at P, denoted $T_{X,P}$, is the set of all lines through P tangent to X. More precisely, given a polynomial $f \in R$, we write $f^{(1)}$ for the linear part of f. That is

$$f^{(1)} := \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(0)x_j. \tag{1.1}$$

Next, given an ideal J of R, we write $J^{(1)} = \langle f^{(1)} : f \in J\rangle$. Then the tangent space to X at P is defined as $\mathbb{V}(\mathbb{I}(X)^{(1)})$. Perhaps one of the most important results concerning tangent spaces is the following theorem relating the tangent space of a variety at a point to the $(\mathcal{O}_{X,P}/\mathfrak{m}_P)k$-vector space $\mathfrak{m}_P/\mathfrak{m}_P^2$ called the *cotangent space* of the local ring $\mathcal{O}_{X,P}$.

Theorem 1.1. *The tangent space $T_{X,P}$ is naturally isomorphic to the vector space of all linear forms on $\mathfrak{m}_P/\mathfrak{m}_P^2$. That is, $T_{X,P} \cong (\mathfrak{m}_P/\mathfrak{m}_P^2)^* = Hom_k(\mathfrak{m}_P/\mathfrak{m}_P^2, k)$.*

So this theorem says that the notion of tangent spaces is dual to that of a cotangent space. We have just seen that our usual notion of differentiation of functions leads to the notion of a tangent space. But note that this characterization of tangent/cotangent space of a variety seems to be dependent on the point P. Ideally, we would like to know that these k-vector spaces are not simply independent vector spaces at various points on X, but rather come from a more global object on X. This leads to the notion of the module of differentials. We refer the reader to either [5] or [7] for a more comprehensive geometric approach to the theory of differentials.

2. Preliminary Results

In this section, we review some results on tensor products which we will then use as a useful tool to verify some important properties of these so called "Kähler differentials".

Proposition 2.1. *Let R be a commutative ring and let I be an ideal of R. Let N be an R-module. Then $R/I \otimes_R N \cong N/IN$ as R-modules.*

Proof. We give a sketch of the proof. See [1] for a more detailed proof. Note $R/I \otimes_R N$ is generated, as an abelian group, by tensors of the form $(r+I) \otimes n = r(1 \otimes n)$ for $r \in R$ and $n \in N$. So the elements $1 \otimes n$ generate $R/I \otimes_R N$ as an R/I-module. Define an R-module map $N \to R/I \otimes_R N$ by $n \mapsto 1 \otimes n$. This map is surjective with kernel IN. Hence, we obtain an R-module homomorphism $f : N/IN \to R/I \otimes_R N$ given by $n+I \mapsto 1 \otimes n$. The claim is that f is an isomorphism. This follows by noting that the map $R/I \times N \to N/IN$ defined by $(r+I, n) \mapsto rn + IN$ is well-defined and R-balanced (or middle linear with respect to R); so by the universal property of tensor products, must factor through $R/I \otimes_R N$. One can then check that this new induced map is the desired inverse to f. □

It is worth nothing that $R/I \otimes_R N$ is naturally an R/I-module. Our next result concerns extension of scalars for free modules. But first, a theorem.

Theorem 2.2. *Let R be a commutative ring and suppose $M_1, \dots, M_n$, and N are R-modules. Then*

$$(M_1 \oplus \cdots \oplus M_k) \otimes_R N \cong (M_1 \otimes_R N) \oplus \cdots \oplus (M_k \otimes_R N).$$

Proof. The proof is similar in flavor to most results concerning tensor product; define an appropriate map on the usual direct products and then use the universal property of tensor product. In our case, it is sufficient to establish the result for the case $n = 2$ and the general case follows by induction. See [1] for a complete proof. □

Corollary 2.3. *Let $f : R \to S$ be a map of commutative rings with $f(1) = 1$. If $N \cong R^{(n)}$ is a free R-module of rank n, then $S \otimes_R N = S^{(n)}$ is a free S-module of rank n.*

Proof. By theorem (2.2), it is sufficient to show that $S \otimes_R R \cong S$ as S-modules. Again, we only give a sketch of the proof and refer the reader to [1] for all the details. First, observe that S may be viewed as an (right) R-module via $sr = sf(r)$ for $s \in S, r \in R$. It is straightforward that the map $S \times R \to S$ defined by $(s, r) \mapsto sr$ is R-balanced. Now apply the universal property to obtain a map $S \otimes_R R \to S$ given by $s \otimes r \mapsto sr$. The map $S \to S \otimes_R R$ given by $s \mapsto s \otimes 1$ can easily be shown to be the inverse. □

We also require the adjoint-associativity of Hom and tensor products.

Proposition 2.4. *Let R and S be commutative rings with unity. Let A and C be R and S-modules respectively and suppose B is an (R, S)-bimodule. Then there is an isomorphism of abelian groups*

$$\rho \ : \ Hom_S(A \otimes_R B, C) \ \xrightarrow{\cong} \ Hom_R(A, Hom_S(B, C)).$$

Proof. Once again, we only give a sketch of the proof and refer the reader to [1] or [4] for all the details. Suppose $\phi : A \otimes_R B \to C$ is given. For a fixed $a \in A$, let $\varphi_a : B \to C$ be defined via $\varphi_a(b) = \phi(a \otimes b)$. Then $\rho(\phi) = \varphi$ is a homomorphism of abelian groups. Conversely, given a homomorphism $\psi : A \to \mathrm{Hom}_S(B, C)$, one defines the map $A \times B \to C$ via $(a, b) \mapsto (\psi(a))(b)$, where we note that $\psi(a) : B \to C$. By the universal property of tensor products, this map factors through $A \otimes_R B$ in the natural way and one can verify that this map is the unique inverse to ρ. $\square$

We end this section with one last result concerning free modules.

Proposition 2.5. *Let R ba a commutative ring and suppose M is an R-module. Then M is a homomorphic image of some free R-module. In other words, every R-module is an image of some free R-module.*

Proof. See [4] for a proof. $\square$

3. Derivations and Kähler Differentials

We begin with the definition of a *derivation*.

Definition 3.1 (Derivation)**.** Let R be a commutative ring and suppose M is an R-module. A *derivation* from R to M is a map $d : R \to M$ satisfying $d(f + g) = d(f) + d(g)$ and $d(fg) = gd(f) + fd(g)$ for all $f, g \in R$. The second property is usually referred to as the Leibniz rule. Note this map makes sense as M is an R-module.

Next, we show that derivations behave very similarly to tensor products. More precisely, we have

Proposition 3.2. *Given a commutative ring R, there exists an R-module Ω_R (and often denoted Ω^1_R) and a derivation $D : R \to \Omega_R$ satisfying the following universal property. For every R-module M and a derivation $d : R \to M$, there exists a unique map $\phi : \Omega_R \to M$ making the following diagram commute*

$$\begin{tikzcd} R \arrow[rr, "D"] \arrow[dr, "d"'] & & \Omega_R \arrow[dl, dashed, "\phi"] \\ & M & \end{tikzcd}$$

Proof. We proceed just as in the case of tensor products. Namely, consider the free R-module on the symbols $D(f)$ for all $f \in R$. Let Ω_R be the quotient of this free R-module by the R-module generated by the usual relations

$$D(f + g) - D(f) - D(g) \qquad \text{and} \qquad D(fg) - gD(f) - fD(g),$$

for all $f, g \in R$. From our construction, the map $D : R \to \Omega_R$ given by $f \mapsto D(f)$ is a derivation. Universality of Ω_R follows from the universal property of free modules (see Chapter 10 of [1]). Lastly, given a derivation $d : R \to M$, define $\phi : \Omega_R \to M$ by $\phi(D(f)) = d(f)$. By construction, $d = \phi \circ D$, and so we are done. $\square$

Remark 3.3. Write $\mathrm{Der}(R, M)$ for the abelian group of derivations from R to M. This is naturally an R-module with multiplication defined by $rd : f \mapsto r(d(f))$ for all $r \in R$ and all derivations d from R into M. It then follows from proposition (3.2) that there is a canonical isomorphism

$$\mathrm{Hom}_R(\Omega_R, M) \cong \mathrm{Der}(R, M),$$

for each R-module M.

Definition 3.4 (Module of Kähler Differentials)**.** The R-module Ω_R of proposition (3.2) is called the *module of Kähler differentials* of R.

Next, we introduce the notion of a *relative differentials*. First, a definition.

Definition 3.5 (S-linear Derivations)**.** Let R and S be commutative rings and suppose $f : S \to R$ is a ring homomorphism; i.e., R is an S-algebra. Suppose M is an R-module. A derivation $d : R \to M$ is said to be *S-linear* if $d(f(s)) = 0$ for all $s \in S$. That is, d is called *S-linear* if d is an S-homomorphism.

The next proposition is analogous to proposition (3.2).

Proposition 3.6. *Suppose R and S are commutative rings and suppose R is an S-algebra. That is, we have a ring homomorphism $f : S \to R$. Then there exists a universal R-module $\Omega_{R/S}$ together with an S-linear derivation $D : R \to \Omega_{R/S}$.*

Proof. This is analogous to the proof of proposition (3.2). Indeed, we always have a universal map $D' : R \to \Omega_R$. This need not be S-linear. So to ensure that this is the case, we quotient out by appropriate relations. Namely, set $\Omega_{R/S}$ to be the quotient of $\{D'(f) : f \in R\}$ by $D'(f+g)-D'(f)-D'(g)$, $D'(fg)-gD'(f)-fD'(g)$, and $D'(f(s))$ for all $s \in S$. The rest is analogous to proof of (3.2). □

Considering the setting of proposition (3.6), the R-module $\Omega_{R/S}$ is called the *module of relative Kähler differentials* of R. Lets consider some examples to help put all this in context. Our first example comes from the theory of curves.

Example 3.7. Let k be a field and suppose C is a curve over k. Then the space of differential forms C (1-forms), denoted Ω_C is the k-vector space generated by the symbols df for all $f \in k(C)$, the function field of C, subject to the relations $f(f+g) = df + dg$, $d(fg) = gdf + fdg$, and $d\alpha = 0$ for all $\alpha \in k$. Elements of Ω_C are usually denoted by ω. It is well-known that there are no non-zero holomorphic differential forms on $\mathbb{P}^1$ (see either [6] or [5] for a full discussion of this fact). Since every genus zero curve can be shown to be isomorphic to $\mathbb{P}^1$, one concludes that there are no non-zero holomorphic differential forms on a genus zero curve.

Example 3.8. Let $k = \mathbb{R}$ and consider the polynomial ring $R = k[x, y]$. Then the usual partial differentiation operator $\frac{\partial}{\partial x}$ from calculus is clearly a derivation from R to itself. In fact, this is $k[y]$-linear and so the module $\mathrm{Der}_{k[y]}(R, R)$ is a free R-module of rank 1 generated by $\frac{\partial}{\partial x}$.

Remark 3.9. Again, note that proposition (3.6) is equivalent to saying that there is a canonical isomorphism, for each R-module M,

$$\mathrm{Hom}_R(\Omega_{R/S}, M) \cong \mathrm{Der}_S(R, M),$$

where $\mathrm{Der}_S(R, M)$ denotes the R-module of S-linear derivations from R to M.

The following lemma will prove fruitful.

Lemma 3.10. *Consider the polynomial ring $R := S[x_1, \dots, x_n]$. Then*

$$\Omega_{R/S} \cong R^{(n)} = \bigoplus_{j=1}^{n} R.$$

Proof. Consider the map $R \to R^{(n)}$ given by $f \mapsto \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$ and note this is an S-linear derivation. By the universal property, we have a map $\phi : \Omega_{R/S} \to R^{(n)}$ given by $df \mapsto \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right)$. We claim that ϕ is an isomorphism. Indeed, note ϕ is surjective since $dx_j \mapsto (0, \dots, 0, 1, 0, \dots, 0)$, where there is a 1 in the j^{th} position. Now consider the map $\psi : R^{(n)} \to \Omega_{R/S}$ given by $(a_1, \dots, a_n) \mapsto \sum_{j=1}^n a_j dx_j$. We wish to show that ψ is the desired inverse to ϕ. Indeed, note

$$\phi(\psi((a_1, \dots, a_n))) = \phi\left(\sum_{j=1}^{n} a_j dx_j\right) = \phi(a_1 dx_1) + \cdots + \phi(a_n dx_n) = (a_1, \dots, a_n).$$

This shows that $\phi \circ \psi = \mathrm{id}_{R^{(n)}}$. Conversely, we wish to show that $\psi \circ \phi = \mathrm{id}_{\Omega_{R/S}}$. That is, we need $\psi(\phi(df)) = df$ for all $f \in R$. But this is equivalent to showing that $df = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j$. Since both sides respect addition and agree on monomials of degree 1, the result follows. $\square$

Remark 3.11. Observe that the above lemma implies that

$$\Omega_{R[x_1, \dots, x_n]/R} \cong \bigoplus_{j=1}^{n} R[x_1, \dots, x_n] dx_j.$$

That is, $\Omega_{R[x_1, \dots, x_n]/R}$ is free on generators $dx_1, \dots, dx_n$. Note the $R[x_1, \dots, x_n]$ is generated as an R-algebra by $x_1, \dots, x_n$ and so $\Omega_{R[x_1, \dots, x_n]/R}$ is generated as an $R[x_1, \dots, x_n]$-module by $dx_1, \dots, dx_n$.

We are now in a position to discuss two very important results concerning the module of relative differentials, which will have great computational implications.

Lemma 3.12. *[First Exact Sequence] Suppose A, B, and C are rings and suppose there exists maps $A \to B \to C$. Then there exists an exact sequence of C-modules.*

$$C \otimes_B \Omega_{B/A} \xrightarrow{f} \Omega_{C/A} \xrightarrow{g} \Omega_{C/B} \longrightarrow 0,$$

where $f(c \otimes db) = cdb$ and $g(dc) = dc$.

Proof. First, note $C \otimes_B \Omega_{B/A}$ is by definition a C-module so the above map is, indeed, a map of C-modules. Surjectivity of g is clear, since g maps generators of $\Omega_{C/A}$ onto the generators of $\Omega_{C/B}$. The only difference is that $\Omega_{C/B}$ has more relations; namely, we must ensure $db = 0$ for all $b \in B$ and this does not affect the generating set of $\Omega_{C/B}$. Lastly, observe that the elements $1 \otimes db$ generate $C \otimes_R \Omega_{B/A}$ as a C-module. But then $f(1 \otimes db) = db$, $b \in B$, and these are precisely the elements in $\ker g$. $\square$

The next lemma is of particular importance to us.

Lemma 3.13. *[Second Exact Sequence] Let R be a ring and suppose R is an S-algebra. Let I is an ideal of R and set $T = R/I$. Then there exists an exact sequence of T-modules*

$$I/I^2 \xrightarrow{d} T \otimes_S \Omega_{R/S} \xrightarrow{D} \Omega_{T/S} \longrightarrow 0,$$

where $D(c \otimes db) = cdb$ and $d(f + I^2) = 1 \otimes df$.

Proof. We begin by noting a few important observations. First, The map d is well-defined. For simplicity, let us write f for $f + I^2$. Suppose we have two coset representatives $f, g \in I$ such that $f - g \in I^2$. Then we may find $h_1, h_2 \in I$ such that $f - g = h_1 h_2$. Applying d gives

$$d(f - g) = 1 \otimes d(h_1 h_2) = 1 \otimes (h_2 dh_1 + h_1 dh_1). \tag{3.1}$$

Now by proposition (2.1) we have an isomorphism

$$T \otimes_R \Omega_{R/S} = R/I \otimes_R \Omega_{R/S} \cong \Omega_{R/S}/I\Omega_{R/S}. \tag{3.2}$$

Recall that the isomorphism in (3.2) is given, again by proposition (2.1), by $(r + I \otimes dr') \mapsto rdr' + I\Omega_{R/S}$. Hence, the right hand side of (3.1) is zero, and we have $d(f - g) = d(f) - d(g) = 0$, proving that d is well-defined.

Further, applying proposition (2.1) once more we see that we have an isomorphism $R/I \otimes_R I \cong I/I^2$. Now as R is an S-algebra, we have the universal derivation $\delta : R \to \Omega_{R/S}$. Consider the restriction of δ to I. Let $\nu := \delta|_I$. It then follows that the map d in consideration is, in fact, the map

$$\mathrm{id}_{R/I} \otimes \nu \ : \ R/I \otimes_R I \ \longrightarrow \ R/I \otimes_R \Omega_{R/S}.$$

Moreover, I/I^2 is clearly a T-module by an earlier remark in the previous section. Also, d is a map of T-modules. Indeed, let $a \in I$ and let $t \in T$ be given. Then

$$d(at) = 1 \otimes (tda + adt) = (1 \otimes tda) + (1 \otimes adt) = 1 \otimes tda = t(1 \otimes a) = td(a),$$

where the third equality follows from a similar argument to the one given in the first paragraph.

We proceed to the proof of the lemma. Note D is clearly surjective by definition of T. So we need to verify that $\Omega_{T/S}$ is the cokernel of d. Indeed, consider $T \otimes_S \Omega_{R/S}$ as a T-module. It is generated by elements of the form dr for $r \in R$ subject to linearity, the Leibniz rule, and $ds = 0$ for all $s \in \operatorname{im} f$, where $f : S \to R$ exhibits R as an S-algebra. Similarly, $\Omega_{T/S} = \Omega_{(R/I)/S}$ is also generated by the same elements, except there are extra relations; namely, every $da = 0$ for each $a \in I$. But these are precisely the images of $d : I/I^2 \to T \otimes_S \Omega_{R/S}$. So $\Omega_{T/S}$ may be obtained from $T \otimes_S \Omega_{R/S}$ by adding these additional relations da for $a \in I$. $\square$

In algebraic geometry we are constantly dealing with various varieties and their respective coordinate rings. As we saw in the introduction the coordinate ring of an variety can be used to give useful information concerning the variety itself. So let us apply the theory we have developed so far to study coordinate rings from the point view of the module of Kähler differentials.

Example 3.14. We use lemma (3.13) to give an explicit construction of the module of Kähler differentials of the coordinate ring of an affine variety. Namely, we compute $\Omega_{R/S}$ in the case where $R = S[x_1, \ldots, x_n]/I$ where $I = \langle f_1, \ldots, f_m \rangle$ represents the ideal of some affine variety X. By corollary (2.3) and lemma (3.10), we immediately have

$$R \otimes_S \Omega_{S[x_1,\ldots,x_n]/S} \cong \bigoplus_{j=1}^{n} R dx_j = R^{(n)}.$$

So by lemma (3.13), we know $\Omega_{R/S}$ is the cokernel of the map

$$\delta : I/I^2 \longrightarrow R^{(n)},$$

where δ is the map d from lemma (3.13). By proposition (2.5), we may find a free R-module $R^{(m)} = \bigoplus_{i=1}^{m} Re_i$ together with a surjection $\phi : R^{(m)} \twoheadrightarrow I/I^2$, where $e_1, \ldots, e_m$ form a basis for $R^{(m)}$ and $\phi(e_i) = f_i + I^2$. Consider the composition

$$R^{(m)} \xrightarrow{\phi} I/I^2 \xrightarrow{\delta} R^{(n)}, \qquad e_i \longmapsto f_i + I^2 \longmapsto df_i.$$

This composition is a map of free R-modules and by the Leibniz rule, we know $df_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}$, for each $i = 1, \ldots, m$. In other words, we have identified $\Omega_{R/S}$ as the cokernel of the Jacobian matrix

$$\mathcal{J} := \begin{bmatrix} \frac{\partial f_1}{\partial dx_1} & \frac{\partial f_1}{\partial dx_2} & \cdots & \frac{\partial f_1}{\partial dx_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial dx_1} & \frac{\partial f_m}{\partial dx_2} & \cdots & \frac{\partial f_m}{\partial dx_n} \end{bmatrix}.$$

For instance, consider the coordinate ring $R = k[x, y, z]/\langle y^2 - x^2 + xyz^2 \rangle$ corresponding to the hypersurface $X \subset \mathbb{A}_k^3$ generated by the ideal $I = \langle y^2 - x^2 + xyz^2 \rangle$. Then $\Omega_{R/k}$ is generated by the symbols dx, dy, and dz modulo the relation $(-2x + yz^2)dx + (2y + xz^2)dy + (2xyz)dz = 0$.

We now show that the cotangent space of a local ring R is naturally isomorphic to the k-vector space $\Omega_{R/k} \otimes_R k$, where k is the residue field of R. Thus, this connects the notion of a cotangent space to that of Kähler differentials.

Theorem 3.15. *Let R be a local ring with maximal ideal $\mathfrak{m}$. Let $k = R/\mathfrak{m}$ be its residue field. Then the map d of lemma (3.13) is an isomorphism of k-vector spaces and so we have $\mathfrak{m}/\mathfrak{m}^2 \cong \Omega_{R/k} \otimes_R k$.*

Proof. By definition, $\Omega_{(R/\mathfrak{m})/k} = \Omega_{k/k} = 0$. So by the Second Exact Sequence, the map $d : \mathfrak{m}/\mathfrak{m}^2 \to \Omega_{R/k} \otimes_R k$ is a surjection. We must show that d is also an injection. From linear algebra, this is equivalent to showing that the dual map of k-vector spaces is surjective. That is, we show that

$$d^* \ : \ \mathrm{Hom}_k(\Omega_{R/k} \otimes_R k, k) \ \longrightarrow \ \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k),$$

is a surjection. By proposition (2.4) and the fact that $\mathrm{Hom}_T(T, T) \cong T$ (both as T-modules and rings) for any commutative ring T with unity, we know

$$\mathrm{Hom}_k(\Omega_{R/k} \otimes_R k, k) \cong \mathrm{Hom}_R(\Omega_{R/k}, \mathrm{Hom}_k(k, k)) \cong \mathrm{Hom}_R(\Omega_{R/k}, k). \tag{3.3}$$

By remark (3.9), the right hand side of (3.3) is canonically isomorphic to $\mathrm{Der}_k(R,k)$. So it is sufficient to show that k-linear derivations from R to k are the same as $\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$. First, observe that if $\delta : R \to k$ is a derivation, then $d^*(\delta) := \delta|_{\mathfrak{m}}$. Also, given $m, m' \in \mathfrak{m}$, the Leibniz rule implies $\delta(mm') = m'\delta(m) + m\delta(m') = 0$ in $R/\mathfrak{m} \cong k$. Now suppose we are given a k-homomorphism $\phi : \mathfrak{m}/\mathfrak{m}^2 \to k$. Note we have a split short exact sequence $0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0$ and so $R \cong R/\mathfrak{m} \oplus \mathfrak{m}$. Given $r \in R$, we have a unique representation $r = \alpha + m$ for some $\alpha \in k$ and $m \in \mathfrak{m}$. Define $\hat{d} : R \to k$ via $\hat{d}(r) = \phi(\overline{m})$, where $\overline{m}$ denotes the image of m in $\mathfrak{m}/\mathfrak{m}^2$. The claim is that $\hat{d}$ is a k-linear derivation of R into k. Indeed, let $r, s \in R$ and write $r = \alpha_r + m_r$ and $s = \alpha_s + m_s$ for unique $\alpha_r, \alpha_s \in k$ and $m_r, m_s \in \mathfrak{m}$. Then

$$\hat{d}(r+s) = \phi(\overline{m_r} + \overline{m_s}) = \phi(\overline{m_r}) + \phi(\overline{m_s}) = \hat{d}(r) + \hat{d}(s).$$

Similarly,

$$\hat{d}(rs) = \phi(\overline{\alpha_r m_s} + \overline{\alpha_s m_r} + \overline{m_r m_s}) = \alpha_s \phi(\overline{m_r}) + \alpha_r \phi(\overline{m_s}),$$

where the last equality follows since $m_r m_s \in \mathfrak{m}^2$, and is therefore zero in $\mathfrak{m}/\mathfrak{m}^2$. On the other hand,

$$s\hat{d}(r) + r\hat{d}(s) = s\phi(\overline{m_r}) + r\phi(\overline{m_s}) = \alpha_s \phi(\overline{m_r}) + \alpha_r \phi(\overline{m_s}).$$

Hence, $\hat{d}$ is indeed a derivation of R into k. Now the association $d^*(\hat{d}) = \phi$ is clear by our earlier remark. This shows that d^* is surjective, as desired. □

References

1. D. S. Dummit, R. M. Foote, *Abstract Algebra, Third Edition*, John Wiley & Sons, Inc., 2004.
2. D. Eisenbud, *Commutative Algebra With a View Towards Algebraic Geometry*, Graduate Texts in Mathematics 150, Springer, 1991.
3. R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics 52, 1977.
4. T. W. Hungerford, *Algebra*, Graduate Texts in Mathematics 73, 1974.
5. I. R. Shafarevich, *Basic Algebraic Geometry 1*, Springer-Verlag, Springer, 1994.
6. J.H. Silverman, *Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics 106, Springer, 2009.
7. R. Vakil, *Foundations of Algebraic Geometry*, June 11, 2013. (`http://math.stanford.edu/~vakil/216blog/FOAGjun1113public.pdf`).

Department of Mathematics, Simon Fraser University, 8888 University Drive, Burnaby, BC, Canada, V5A 1S6

E-mail address: `naa15@sfu.ca`